Wigner surmise for high-order level spacing distributions of chaotic systems

A. Y. Abul-Magd
Department of Mathematics and Computer Science, Faculty of Science, UAE University, P.O. Box 17551, Al-Ain, United Arab Emirates

M. H. Simbel
Department of Physics, Faculty of Science, Zagazig University, Zagazig, Egypt

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We suggest an extension of the Wigner surmise for the nearest-neighbor-spacing distribution of energy levels of chaotic systems to include the nth-order spacing distributions. The main assumption is that the conditional probability density of occurrence of a level at a given distance from a fixed level, provided that this distance contains n levels, is expressed in terms of the $(n+1)$th power of corresponding probability for a distance containing no levels. At large spacings, the nth-order level distributions are assumed to have a Gaussian shape as in the cases covered by the Wigner surmise. The expressions obtained are in good agreement with the results of numerical calculation by means of the random matrix theory for the three universal classes of symmetry. [S1063-651X(99)05210-1]

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I. INTRODUCTION

The random matrix theory [1–3] presents a natural framework for describing fluctuation properties of spectra of quantum systems, whose classical dynamics is chaotic. The theory considers three types of random matrix ensembles, namely, the Gaussian orthogonal ensemble (GOE), the Gaussian unitary ensemble (GUE), and the Gaussian symplectic ensemble (GSE). The GOE is appropriate for systems with time reversal symmetry and integral spin, the GUE applies to systems without time reversal symmetry, and the GSE is applicable to systems with time reversal symmetry and half-integral spin. The symmetry properties define the degree of level repulsion in the energy spectrum of the ensemble [4]. The nearest-neighbor-spacing distribution $P_\beta(s)\propto s^\beta$ for $s\to 0$, where $\beta=1$, 2, and 4 for the GOE, GUE, and GSE, respectively. The Gaussian distribution of the Hamiltonian matrix elements suggests a Gaussian falloff of $P_\beta(s)$ at large s. A reasonable representation for $P(s)$ is given by the so-called Wigner surmise

$$P_\beta(s)=A_\beta s^\beta e^{-B_\beta s^2}, \tag{1}$$

where the parameters A_β and B_β are obtained from the normalization condition and the requirement that the mean spacing is unity. They are specifically given by $A_1=\pi/2$, $B_1=\pi/4$ (GOE), $A_2=32/\pi^2$, $B_2=4/\pi$ (GUE), and $A_4=262\,144/729\pi^3$, $B_4=64/9\pi$ (GSE), respectively. The Wigner surmise is an exact result only for ensembles of 2×2 matrices. It nevertheless provides an excellent approximation for the exact numerical calculations obtained for the higher-order matrices, and has indeed been useful in the analysis of experimental and numerical-experimental results involving chaotic systems. It has also provided a basis for numerous attempts to evaluate the nearest-neighbor-spacing distributions of levels for a system with mixed ordered-chaotic classical dynamics, e.g., [5–12].

In this paper, we generalize the Wigner surmise for $P_\beta(s)$ to the nth-order spacing distribution $p_\beta(n,s)$ for chaotic systems belonging to the above-mentioned three classes of symmetry. The term $p_\beta(n,s)\,ds$ represents the probability that an interval of length s which starts at an arbitrarily chosen level contains exactly n levels and that the next $(n+1)$th level is in the interval $[s,s+ds]$. In this notation, the nearest-neighbor-spacing distribution $P_\beta(s)$ is expressed as $p_\beta(0,s)$. Exact expressions for these functions as well as

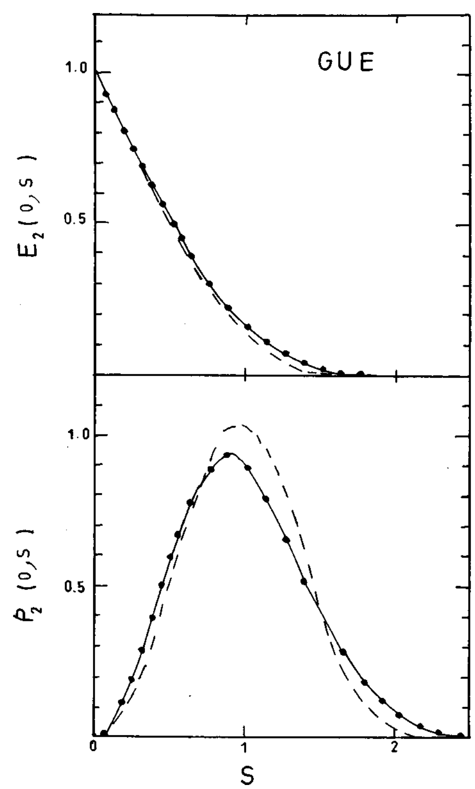

FIG. 1. Nearest-neighbor-spacing distribution $p_2(0,s)$ and gap distribution $E_2(0,s)$ for levels of a GUE. The solid curves are calculated using the Wigner surmise (1), the dotted curves are for the Brody-like distribution (4), and the dots are the numerical values reported in Mehta's book [1].

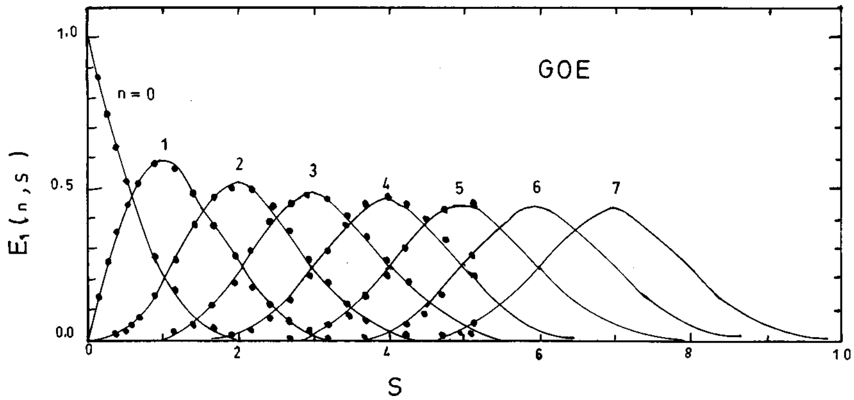

FIG. 2. Probabilities $E_1(n,s)$ of having n levels of a GOE in an interval s. The solid curves are obtained by applying the proposed generalization of the Wigner surmise. The dots are the numerical values reported in Mehta's book [1].

tables for the related distributions are available, e.g., in [1]. High-order spacing distributions have recently been considered for quasi-integrable billiards [13], as well as for systems in transition between regularity and chaos [14]. In Ref. [14], the nth-order spacing distributions are obtained by applying a statistical approach similar to that used in deriving the Brody distribution [15] that assumes a fractional power for the level repulsion. Section II reviews these statistical arguments, showing that they are not successful for repulsion powers $\beta>1$ and can reproduce only the small-spacing section of the level distributions for the GUE, for example. We apply the statistical approach only to evaluate the small s behavior of $p_\beta(n,s)$. We assume that the probability of occurrence of $n+1$ levels within a distance s from a given level is related to the probability of occurrence of a single level raised to the power $n+1$. We then obtain a Wigner-like surmise for $p_\beta(n,s)$ by assuming a Gaussian falloff at large s. The summary and conclusions of this work are given in Sec. III.

II. HIGH-ORDER LEVEL SPACING DISTRIBUTION

Wigner [16] suggested a simple probabilistic approach to calculating the nearest-neighbor-spacing distribution. He obtains

$$P_\beta(s)=r_\beta(s)\exp\left[-\int_s^\infty r_\beta(x)dx\right], \tag{2}$$

where $r_\beta(s)$ is the level-repulsion function defined so that $r_\beta(s)ds$ is the conditional probability that, given a level at energy E, there is one level in the interval ds at a distance s and no levels in the interval $(E,E+s)$. The requirement that $P_\beta(s)$ behave at small s according to a power law implies that

$$r_\beta(s)\propto s^\beta \quad \text{for } s\to 0. \tag{3}$$

We stress here that the previous relation is an asymptotic relation and cannot simply be used to define the dependence of $r_\beta(s)$ in the whole interval of $0<s<\infty$. Only in the case of the GOE when $\beta=1$ does the choice $r_\beta(s)\propto s^\beta$ work to reproduce the Wigner surmise for this particular case. Brody's formula for mixed systems [15] is obtained by setting $r_\beta(s)\propto s^\beta$ for all values of s and, without a theoretical foundation, allowing β to vary between 0 and 1 to interpolate between the Poisson and Wigner distributions. The success of Brody's formula in the analysis of empirical spacing distribution is not sufficient to justify the power law for the level repulsion function, especially when other distributions that have stronger theoretical foundations can fit the data equally well if not better [12].

The choice of $r_\beta(s)\propto s^\beta$ will clearly produce wrong results for the spacing distributions of the GUE and GSE, where $\beta=2$ and 4, respectively. We demonstrate this on the example of GUE. Assuming that $r_2(s)=cs^2$, where c is a constant, and substituting into Eq. (2), we obtain

$$P_2(s)=cs^2\exp\left[-\frac{1}{3}cs^3\right], \quad c=\frac{1}{9}\left[\Gamma\left(\frac{1}{3}\right)\right]^3. \tag{4}$$

Figure 1 shows a comparison between the prediction of the Brody-like distribution (4), shown by the dashed line, the Wigner surmise for the GUE, shown by the solid line, and the exact numerical results, taken from Mehta's book [1] and

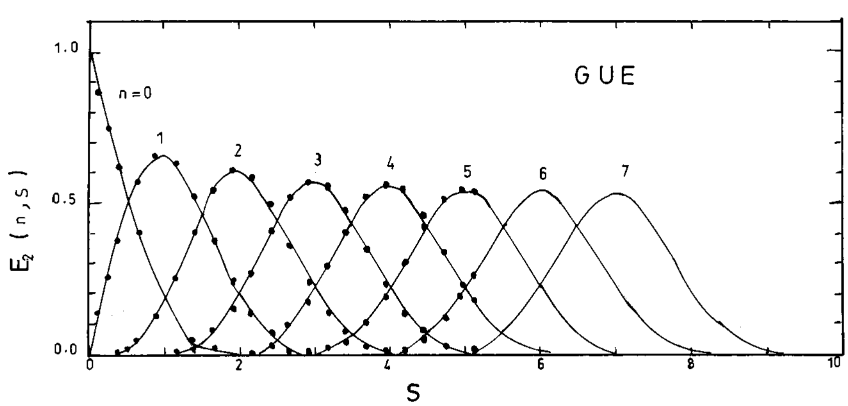

FIG. 3. Probabilities $E_2(n,s)$ of having n levels of a GUE in an interval s. The solid curves are obtained by applying the proposed generalization of the Wigner surmise. The dots are the numerical values reported in Mehta's book [1].

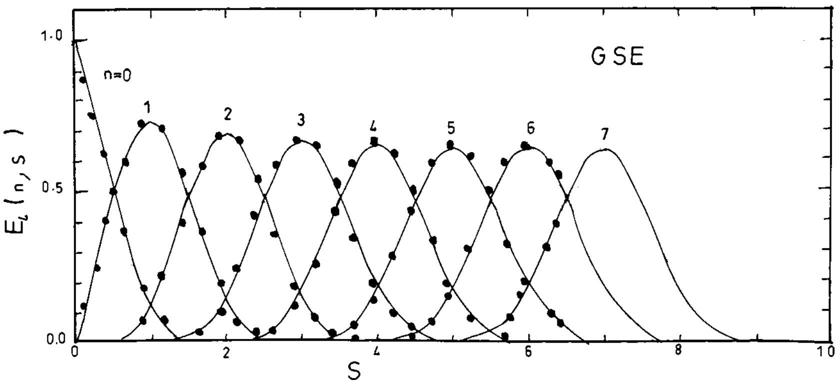

FIG. 4. Probabilities $E_4(n,s)$ of having n levels of a GSE in an interval s. The solid curves are obtained by applying the proposed generalization of the Wigner surmise. The dots are the numerical values reported in Mehta's book [1].

shown in the figures as dots. The figure also shows the same comparison for the gap distribution function defined by

$$E_\beta(0,s)=\int_s^\infty dy\int_y^\infty P_\beta(x)dx. \tag{5}$$

It is clear from the figure that, while the predictions of the Wigner surmise practically coincide with the exact results, the agreement of the latter distributions with the Brody-like formulas is reasonable only for spacings much less than unity. We expect the disagreement to be even stronger in the case of the GSE where the exponential function in Brody-like distribution has an argument proportional to s^5.

Engel, Main, and Wanner [14] applied Wigner's approach to obtain the following expression for the nth-order spacing distribution:

$$p_\beta(n,s)=r_\beta(n,s)\times\exp\left[-\int_0^s r_\beta(n,s)ds\right]\times\int_0^s p_\beta(n-1,x)\exp\left[\int_0^x r_\beta(n,y)dy\right]dx, \tag{6}$$

where $r_\beta(n,s)ds$ is the conditional probability that a new $[(n+1)\text{th}]$ level occurs in an interval ds at a distance s from an arbitrary chosen level, provided that this distance contains exactly n levels. In the following, we apply this result to deduce the small-spacing behavior for the nth-order spacing distribution $p_\beta(n,s)$ and derive the full distribution by assuming a Gaussian falloff at large spacing.

Our main assumption is that the occurrence of consecutive levels in a chaotic system is a random process. This allows us to express the conditional probability $r_\beta(n,s)$ of occurrence of the $(n+1)$th level at a distance s in terms of the conditional probability $r_\beta(s)$ of occurrence of a single level raised to the power $n+1$, at least for small values of s. Therefore, taking Eq. (3) into account, we propose the following ansatz:

$$r_\beta(n,s)\propto s^{(n+1)\beta}\quad\text{for}\quad s\to 0. \tag{7}$$

Accordingly, we rewrite Eq. (6) in the domain of small s as

$$p_\beta(n,s)=c_n s^{(n+1)\beta}\int_0^s p_\beta(n-1,x)dx, \tag{8}$$

where c_n is a constant. Solving this equation by induction, we obtain

$$p_\beta(n,s)\propto s^{\alpha_{\beta,n}}\quad\text{for}\quad s\to 0, \tag{9}$$

where

$$\alpha_{\beta,n}=n+\frac{(n+1)(n+2)}{2}\beta. \tag{10}$$

Equation (9) agrees with the characteristic form of the high-order level repulsion effect for small spacings that has already been put forward by Porter using a completely different argument [17]. This can be regarded as a confirmation of the old result and the proposed line of ansatz (7).

Finally, assuming that a Gaussian law governs the falloff of the function $p_\beta(n,s)$ at large s, we obtain the following generalization of the Wigner surmise:

$$p_\beta(n,s)=A_{\beta,n}s^{\alpha_{\beta,n}}\exp(-B_{\beta,n}s^2), \tag{11}$$

where the constants $A_{\beta,n}$ and $B_{\beta,n}$ are obtained from the conditions that $\int_0^\infty p_\beta(n,s)ds=1$ and $\int_0^\infty sP_\beta(n,s)ds=n+1$. These conditions yield

$$A_{\beta,n}=2B_{\beta,n}^{(\alpha_{\beta,n}+1)/2}\Bigg/\Gamma\left(\frac{\alpha_{\beta,n}+1}{2}\right)$$

and

$$B_{\beta,n}=\left[\Gamma\left(\frac{\alpha_{\beta,n}}{2}+1\right)\Bigg/\left\{(n+1)\Gamma\left(\frac{\alpha_{\beta,n}+1}{2}\right)\right\}\right]^2. \tag{12}$$

The probability $E_\beta(n,s)$ that the interval s contains n levels can be expressed by means of Eq. (5) as a double integral of a combination of the functions $p_\beta(n,s)$, which is reduced, after changing the integration order, to the form

$$E_\beta(n,s)=\int_s^\infty(x-s)[p_\beta(n,x)-2p_\beta(n-1,x)+p_\beta(n-2,x)]dx. \tag{13}$$

This equation is valid for $n\geqslant 2$ but can also be applied for the lower values of n if one defines $p_\beta(j,s)=0$ for $j=-1$ and -2.

We have applied Eqs. (11) and (13) to calculate the distributions $E_\beta(n,s)$ with values of n ranging from 0 to 7 for the GOE, GUE, and GSE. The results of calculation are compared with the exact values reported by Mehta [1] in Figs. 2, 3, and 4, respectively. As follows from these figures, the agreement is perfect.

III. SUMMARY AND CONCLUSION

The Wigner surmise presents a simple and accurate representation for the nearest-neighbor-spacing distribution of levels for chaotic systems. The present work proposes a generalization of the surmise to describe high-order spacing distribution. We use a statistical approach that has been successful in calculating the lowest-order distributions for systems in transition between the Poisson and GOE statistics. We assume that the nth-order level repulsion function is proportional to the $(n+1)$th power of the zero-order function, at least in the small-spacing domain. The falloff of the distribution at large spacing is assumed to follow a Gaussian law. We then obtain an expression for the nth-order spacing distribution which is reduced to the Wigner surmise by setting $n=0$. This expression is found to be in excellent agreement with the exact results of the random matrix theory.

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